A Moving Horizon Convex Relaxation for Mobile Sensor Network Localization

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Abstract—In mobile sensor network localization problems we seek to estimate the position of the mobile sensor nodes by using a subset of pair-wise range measurements (among the nodes and with mobile anchors). When the sensor nodes are static, convex relaxations have been shown to provide a remarkably accurate approximate solution to this NP-hard estimation problem. In this paper, we propose a novel convex relaxation to tackle the more challenging dynamic case and we develop a moving horizon convex estimator based on a maximum a posteriori (MAP) formulation. The resulting estimator is then compared to standard approaches seeking for approximate solutions. However, these methods are often ad-hoc. Moreover, the estimation task is complicated by the fact that the sensor nodes are moving and the estimation problem is known as mobile sensor network localization.

Our problem concerns estimating the position of a number of mobile sensor nodes at specific discrete time steps, via a subset of their pair-wise distances and a subset of their distances with a set of mobile anchors. This estimation problem is tackled with standard estimation techniques, such as extended and unscented Kalman filters [3], [4] and particle filters [5]–[7] (and references therein). In the case of highly connected networks, multi-dimensional scaling has been extended to mobile nodes, for example in [8]. In this paper, we use moving horizon estimators [9]–[11] and we extend the convex relaxation techniques that have shown their remarkable performance in static scenarios. Our goal is to understand whether in the dynamic case, we can obtain a similar gain in accuracy with respect to more traditional techniques. Preliminary results will answer this question both affirmatively and negatively: it is true that we achieve a significant improvement in convergence speed, but the steady-state accuracy suffers from the relaxed nature of the estimator more than in the static case.

Relation to previous work. The nonlinear estimation problem arising from mobile sensor network localization has been

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I. INTRODUCTION

Our problem stems both from its practical relevance and from its theoretical challenges. In particular, the problem at hand is NP-hard, which has lead (at least in the static case) to the development of a rich variety of heuristic approaches seeking for approximate solutions. However, these methods are often ad-hoc. Moreover, the estimation task is complicated by the fact that the sensor nodes are moving and the pair-wise distance measurements alone might be too few per time step to localize the sensors uniquely. This makes the characterization of the dynamical model of the sensor nodes a necessity, as well as the design of dynamical filters.

In this paper, we propose a moving horizon convex estimator that is based on a suitable convex relaxation of the maximum a posteriori (MAP) formulation of the mobile network localization problem. This convex relaxation is an extension of the ones used in static problems with remarkable performance, e.g., [1], [2] and references therein. In particular, such extension is twofold. On one side, we relax the moving window estimation problem; on the other side, we also relax the term that summarizes the past information (outside the estimation window), i.e., the so-called arrival cost. As we show, our relaxation leads to a computational complexity comparable with the one of extended and unscented Kalman filters, yet preliminary results indicate its improved convergence speed and, when used together with a gradient-based refinement, similar steady-state accuracy.

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II. PRELIMINARIES

We consider a network of \( n \) mobile wireless sensor nodes with computation and communication capabilities, living in a \( \mathbb{D} \)-dimensional space. We denote the set of all nodes \( \mathcal{N} = \{1, \ldots, n\} \). Let \( x_{i,\tau} \in \mathbb{R}^{\mathcal{D}} \) be the position vector of the \( i \)-th sensor node at the discrete time step \( \tau \in \mathbb{N} \), while let \( v_{i,\tau} \in \mathbb{R}^{\mathcal{D}} \) be its velocity vector at the same discrete time step. Let \( X_{\tau} = [x_{1,\tau}^\top, \ldots, x_{n,\tau}^\top] \in \mathbb{R}^{\mathcal{D} \times n} \) and \( V_{\tau} = [v_{1,\tau}^\top, \ldots, v_{n,\tau}^\top] \in \mathbb{R}^{\mathcal{D} \times n} \) be the matrix collecting the position vectors and velocity vectors, respectively. In addition to the mobile sensor nodes, we consider \( m \) mobile anchor nodes (whose position at each time step is known, e.g., via GPS) and we let \( a_{k,\tau} \), for \( k \in \{1, \ldots, m\} \), be their position in \( \mathbb{R}^{\mathcal{D}} \), and \( A_{\tau} = [a_{1,\tau}, \ldots, a_{m,\tau}] \). We assume that a model is given for the dynamics of the sensor nodes; in particular, we let the sensor nodes evolve via the known linear time-invariant dynamical system

\[
s_{i,\tau+1} = \Phi_s s_{i,\tau} + w_{i,\tau}, \quad \text{for all } i \in \mathcal{N},
\]

where \( s_{i,\tau} := (x_{i,\tau}^\top, v_{i,\tau}^\top)^\top \) is the state vector, \( \Phi_s \in \mathbb{R}^{2\mathcal{D} \times 2\mathcal{D}} \) is the state matrix, and \( w_{i,\tau} \in \mathbb{R}^{2\mathcal{D}} \) is the process noise. We define \( S_{\tau} = [s_{1,\tau}, \ldots, s_{n,\tau}] \) and \( W_{\tau} = [w_{1,\tau}, \ldots, w_{n,\tau}] \) as the matrices collecting the state and the process noise terms, respectively. Since \( W_{\tau} \) is a random variable drawn from a given probability density function (PDF), the dynamics (1) can be equivalently expressed as a discrete-time Markov process with conditional probability \( p_{S_{\tau+1}}(S_{\tau+1} | S_{\tau}) \).
We consider line-of-sight conditions between the nodes and we assume that some pairs of sensor nodes \((i,j) \in \mathcal{E}_T\) have access to noisy range measurements at time \(\tau\) as
\[
r_{i,j,\tau} = d_{i,j,\tau} + \nu_{i,j,\tau},
\]
where \(d_{i,j,\tau} = ||x_{i,\tau} - x_{j,\tau}||\) is the noise-free Euclidean distance and \(\nu_{i,j,\tau}\) is an additive noise term with known probability distribution. We call \(p_{i,j}(d_{i,j,\tau}(x_{i,\tau}, x_{j,\tau})|r_{i,j,\tau})\) the inter-sensor sensing PDF, indicating explicitly the dependence of \(d_{i,j,\tau}\) on the sensor node positions \((x_{i,\tau}, x_{j,\tau})\), and call \(r_{\tau}\) the stacked vector of measurements \(r_{i,j,\tau}\).

In addition, we consider that some sensors also have access to range measurements with some mobile anchor nodes as
\[
z_{i,k,\tau} = e_{i,k,\tau} + \mu_{i,k,\tau}, \quad (i,k) \in \mathcal{E}_{a,\tau}
\]
where \(e_{i,k,\tau} = ||x_{i,\tau} - a_{k,\tau}||\) is the noise-free Euclidean distance and \(\mu_{i,k,\tau}\) is an additive noise term with known probability distribution. We denote as \(p_{i,k,a}(e_{i,k,\tau}(x_{i,\tau}, a_{k,\tau})|z_{i,k,\tau})\) the anchor-sensor sensing PDF and as \(z_{\tau}\) the stacked vector of measurements \(z_{i,k,\tau}\).

At time \(\tilde{\tau}\), the problem of interest can be described as estimating the entire state sequence \((S_0, \ldots, S_{\tilde{\tau}})\), here indicated with \(\{S\}_{0}^{\tilde{\tau}}\), given the dynamics (1), the available pair-wise measurements (2)-(3) for \(\tau = 1, \ldots, \tilde{\tau}\), and a PDF of the initial condition \(p_{S_0}(S_0)\). This can be formulated as the following Bayesian maximum a posteriori (MAP) estimation problem
\[
\{S_{\text{MAP}}\}^{\tilde{\tau}} \in \arg \max \{S\}_{0}^{\tilde{\tau}} p(S_0^{\tilde{\tau}}|\{(r, z)\}_{1}^{\tilde{\tau}}).
\]
Using the Markov property of (1), Bayes’ rule for conditional probability, and the properties of logarithms, we can write arg max \(\{S\}_{0}^{\tilde{\tau}} p(S_0^{\tilde{\tau}}|\{(r, z)\}_{1}^{\tilde{\tau}})\) as
\[
\arg \max \{S\}_{0}^{\tilde{\tau}} \sum_{\tau=1}^{\tilde{\tau}} \left[ \sum_{(i,j) \in \mathcal{E}} \ln p_{i,j}(d_{i,j,\tau}(x_{i,\tau}, x_{j,\tau})|r_{i,j,\tau}) + \sum_{(i,k) \in \mathcal{E}_{a,\tau}} \ln p_{i,k,a}(e_{i,k,\tau}(x_{i,\tau}, a_{k,\tau})|z_{i,k,\tau}) \right] + \sum_{\tau=0}^{\tilde{\tau}-1} \sum_{\tau=0}^{\tilde{\tau}-1} \ln p_{S}(S_{\tau+1}|S_\tau) + \ln p_{S_0}(S_0).
\]
Let us now introduce the Gaussian noise assumption.

Assumption 1: The measurement noise terms \(\nu_{i,j}\) and \(\mu_{i,k}\) drawn from a Gaussian distribution as \(\nu_{i,j} \sim \mathcal{N}(0, \sigma^2_{i,j})\), and \(\mu_{i,k} \sim \mathcal{N}(0, \sigma^2_{i,k,a})\), respectively. The process noise is drawn from a Gaussian distribution as \(w_{i,\tau} \sim \mathcal{N}(0, \Sigma_w)\). The initial condition is drawn from a Gaussian distribution as \(s_{0,0} \sim \mathcal{N}(0, \Pi_{s_0})\).

Using Assumption 1 we can rewrite the right hand term of (4), and thus the MAP estimation problem, as
\[
\{S_{\text{MAP}}\}_{0}^{\tilde{\tau}} \in \arg \min \{S\}_{0}^{\tilde{\tau}} \sum_{\tau=1}^{\tilde{\tau}} f_{r,\tau}(S_\tau) + \sum_{\tau=0}^{\tilde{\tau}-1} f_{S}(S_{\tau+1}, S_\tau) + f_{S_0}(S_0), \quad \text{(5)}
\]
where
\[
f_{S_0}(S_0) := \sum_{i \in \mathcal{V}} ||s_{i,0} - \bar{s}_{i,0}||^2_{\Pi_{s_{i,0}}^{-1}}.
\]
\[
f_{S}(S_{\tau+1}, S_\tau) := \sum_{(i,j) \in \mathcal{E}} ||s_{i,\tau+1} - \Phi_{s} s_{i,\tau}||^2_{\Sigma_{s}^{-1}}
\]
and
\[
f_{r,\tau}(S_\tau) := \sum_{(i,j) \in \mathcal{E}} ||d_{i,j,\tau}(x_{i,\tau}, x_{j,\tau}) - r_{i,j,\tau}||^2_{\sigma^2_{i,j}} + \sum_{(i,k) \in \mathcal{E}_{a,\tau}} ||e_{i,k,\tau}(x_{i,\tau}, a_{k,\tau}) - z_{i,k,\tau}||^2_{\sigma^2_{i,k,a}}.
\]

### III. Problem Statement

In order to solve the MAP problem (5), we need to keep in memory all the measurements from \(\tau = 1\) till \(\tau = \tilde{\tau}\), and the size of the problem grows in time. To overcome this issue, we employ a moving horizon strategy, that is, we formulate the MAP estimator using a moving, but fixed-size, estimation window and approximate the information outside the window. Considering a fixed moving window \(T \geq 1\), the problem that we want to solve in this paper can be written as

\[
\{S_{\text{MAP},T}\}_{\tau_0}^{\tilde{\tau}} \in \arg \min \{S\}_{\tau_0}^{\tilde{\tau}} \sum_{\tau=\tau_0+1}^{\tilde{\tau}} f_{r,\tau}(S_\tau) + \sum_{\tau=\tau_0}^{\tilde{\tau}-1} f_{S}(S_{\tau+1}, S_\tau) + f_{S_0}(S_0), \quad \text{(6)}
\]
where \(\bar{\tau}_0 := \max\{\tilde{\tau} - T, 0\}\), and \(f_{S_0}(S_0)\) is a given function that approximates the influence of the state before \(\bar{\tau}_0\), i.e., an approximate arrival cost.

In this paper, first we discuss a possible choice for \(f_{S_0}(S_0)\), then, since (6) is a non-convex problem, we will make use of a properly defined convex relaxation to massage it and devise efficient and accurate approximate solutions.

### IV. Approximate Arrival Cost

The choice of \(f_{S_0}(S_0)\) is a delicate one and it influences the stability of the MAP estimator. The safest choice is to put \(f_{S_0}(S_0) = 0\); in this way, however, the information carried by the range measurements before the time window is discarded. Another popular choice is to use the recursion
\[
f_{S_0}(S_0) = ||\text{vec}(S_0 - \bar{S}_0)||^2_{\Pi_{S_0}^{-1}}.
\]
where \(\bar{S}_0\) is the estimation of \(S_0\) computed by (6) at the time step \(\bar{\tau} - 1\), and \(\Pi_{S_0}\) is its variance, approximated via a suitable filter, e.g., the extended Kalman filter. In this paper, we take the latter choice, which entails the following recursive estimation for the covariance [9]
\[
\Pi_{S_0} = \text{blkd}[[\Pi_{s_0}]]
\]
\[
\Pi_{S_0} = (I - K_{S_0} H_{S_0})(\text{blkd}[\Phi_{s}]\Pi_{S_0}\text{blkd}[\Phi_{s}^T] + \text{blkd}[\Sigma_w]),
\]
where \(\text{blkd}[;]\) is the block diagonal operator, \(K_{S_0}\) is the Kalman gain, and \(H_{S_0}\) is the gradient of the stacked measurement equations (2)-(3) with respect to \(S\) at the point \(S_{\tau_0}\).

### V. Convex Relaxation

To derive the mentioned convex relaxation of the MAP estimator (6), several steps are needed. First of all, we introduce the new variables \(Y_{\tau} = X_{\tau}^T X_{\tau}, \delta_{i,j,\tau} = d_{i,j,\tau}^2, \epsilon_{i,k,\tau} = e_{i,k,\tau}^2\), and we collect the \(d_{i,j,\tau}, \epsilon_{i,k,\tau}, \delta_{i,j,\tau}, \epsilon_{i,k,\tau}\) scalar variables into the stacked vectors \(d_{\tau}, \epsilon_{\tau}, \delta_{\tau}, \epsilon_{\tau}\). Second, we rewrite the
cost $f_r(x(S_r))$ of the MAP estimator as dependent on $(d_r,e_r,\delta_r,\epsilon_r)$, considered as independent variables, as
\[
\dot{f}_{r,x}(d_r, e_r, \delta_r, \epsilon_r) := \sum_{(i,j) \in E_r} \sigma_{i,j}^{-2}(d_{i,j} - 2d_{i,j}r_{i,j}) + \sum_{(i,k) \in E_{\delta,r}} \sigma_{i,k}^{-2}(\epsilon_{i,k} - 2\epsilon_{i,k}z_{i,k}).
\]

Third, we re-introduce the dependencies of $(d_r, e_r)$ on $X_r$ and on $(\delta_r, \epsilon_r)$ by considering the following constrained optimization
\[
\text{minimize } J(\{S_{\tau_0}\}^{\bar{V}}_r, (Y, \delta, e, d, e)_{\tau_0})_{\tau_0+1} \\
\text{subject to } Y_{ii} + Y_{jj} - 2Y_{ij} = \delta_{i,j},
\delta_{i,j} = d_{i,j}^2, d_{i,j} \geq 0, \quad \forall (i, j) \in E_r\tag{7a}
\]
\[
Y_{ii} - 2x_{i}^T a_{k_{i}} + ||a_{k_{i}}||^2 = \epsilon_{i,k},
\epsilon_{i,k} = e_{i,k} + \epsilon_{i,k}^2, e_{i,k} \geq 0, \quad \forall (i, k) \in E_{a,r}\tag{7b}
\]
\[
Y_r = X_r^T X_r.\tag{7c}
\]

where $J(\{S_{\tau_0}\}^{\bar{V}}_r, (Y, \delta, e, d, e)_{\tau_0})_{\tau_0+1} := \sum_{\tau=\tau_0}^{\bar{V}} \dot{f}_{r,x}(d_r, e_r, \delta_r, \epsilon_r) + \sum_{\tau=\tau_0}^{\bar{V}-1} f_S(S_{\tau+1}, S_r) + f_{S0}(S_{\tau_0}).$

The problem (7) is equivalent to (6): the constraints in the problem (7) have both the scope of imposing the pair-wise distance relations and of enforcing the change of variables (in fact, without the constraints, all the variables would be independent of each other). In the new variables, the cost function is a convex function, however the constraints of (7) still define a non-convex set. Nonetheless, we can massage the constraints by using Schur complements and propose the following convex relaxation
\[
\text{minimize } J(\{S_{\tau_0}\}^{\bar{V}}_r, (Y, \delta, e, d, e)_{\tau_0})_{\tau_0+1} \\
\text{subject to } Y_{ii} + Y_{jj} - 2Y_{ij} = \delta_{i,j},
\delta_{i,j} = d_{i,j}^2, d_{i,j} \geq 0, \quad \forall (i, j) \in E_r\tag{8a}
\]
\[
Y_{ii} - 2x_{i}^T a_{k_{i}} + ||a_{k_{i}}||^2 = \epsilon_{i,k},
\epsilon_{i,k} = e_{i,k} + \epsilon_{i,k}^2, e_{i,k} \geq 0, \quad \forall (i, k) \in E_{a,r}\tag{8b}
\]
\[
\text{subject to } Y_r = X_r^T X_r.\tag{8c}
\]

This convex problem can be now further expressed as a semi-definite program (SDP) using its epigraph form. Introduce the cost $J_{\text{epi}}(\{S_{\tau_0}\}^{\bar{V}}_r, (Y, \delta, e, d, e, \beta)_{\tau_0})_{\tau_0+1} := \sum_{\tau=\tau_0}^{\bar{V}} \dot{f}_{r,x}(d_r, e_r, \delta_r, \epsilon_r) + \sum_{i \in \mathcal{V}} \beta_{i,\tau} + \gamma_{\tau_0},$

where the scalar $\gamma_{\tau_0}$ and the vectors $\{\beta_{i,\tau}\}_{\tau_0+1}$ are slack variables. The problem (8) can then be written in the equivalent form
\[
\text{minimize } J_{\text{epi}}(\{S_{\tau_0}\}^{\bar{V}}_r, (Y, \delta, e, d, e, \beta)_{\tau_0})_{\tau_0+1} \\
\text{subject to } \forall \tau = \tau_0 + 1, \ldots, \bar{V},\tag{8a}, (8b), (8c)
\]
\[
\text{for } \tau = \tau_0 : \quad \vec{S}_{\tau} - \hat{S}_{\tau} \in S^2_{+}^{D_{n+1}},\tag{9a}
\]
\[
\text{for } \tau = \tau_0 + 1, \ldots, \bar{V} - 1 : \quad \Sigma_{w_{\tau}} (s_{\tau,i-1} - \Phi_{s_{i-1},\tau})^T \beta_{i,\tau+1} \geq 0, \quad \forall i \in \mathcal{V},\tag{9b}
\]

Problem (8) and (9) are equivalent and are both rank relaxations of the original receding horizon MAP estimator (6).

VI. AN EDGE-BASED CONVEX RELAXATION

Given the network coupling constraints (8c) and (9a), the solution of (9) has a computational complexity of $O(T^3 \max\{n^3, E^3\})$, with $E$ the average number of edges in the window $T$. To reduce this complexity we adopt a further relaxation of (9) by applying the constraints on the edges only (as done for example in [12]), i.e., we substitute (8c) with
\[
\begin{bmatrix}
I_{D} & x_{i,\tau} & x_{j,\tau} \\
x_{i,\tau}^T & Y_{ii,\tau} & Y_{ij,\tau} \\
x_{j,\tau}^T & Y_{ij,\tau} & Y_{jj,\tau}
\end{bmatrix} \in S^2_{+}^{D+2}, \quad \forall (i, j) \in E_r.\tag{10}
\]

and (9a) with
\[
\begin{bmatrix}
\Pi_{i,i,\tau} & \Pi_{i,j,\tau} \\
\Pi_{i,j,\tau}^T & S_{i,\tau}, S_{j,\tau} - \hat{s}_{i,\tau}, \hat{s}_{j,\tau}, \gamma_{\tau_0}
\end{bmatrix} \in \mathbb{S}_+^{2D+1}, \quad \forall (i, j) \in E_r.\tag{11}
\]

where the $\Pi_{i,i,\tau}$, $\Pi_{i,j,\tau}$, and $\Pi_{i,j,\tau}^T$ are the auto-covariance and cross-covariance matrices of sensor node $i$ and $j$. By doing these substitutions, we consider the following relaxation of (9):

\text{Edge-based moving horizon relaxation:}

\[
\text{minimize } J_{\text{epi}}(\{S_{\tau_0}\}^{\bar{V}}_r, (Y, \delta, e, d, e, \beta)_{\tau_0})_{\tau_0+1} \\
\text{subject to } \forall \tau = \tau_0 + 1, \ldots, \bar{V},\tag{12a}
\]

This relaxation, has now a computational complexity lower bounded by $O(T^3 n^2 E)$ (and upper bounded by the one of (9)) and it is more suitable for on-line computations. We note that standard extended and unscetced Kalman filters have compu-
tional complexity\(^1\) of \(O(\max\{n^3, E_r^3\})\) and \(O((n + E_r)^3)\), respectively (here \(E_r\) is the number of edges at time \(\tau\)). This means that the estimator (12) has comparable computational complexity in terms of the network variables \(n\) and \(E_r\).

VII. PRELIMINARY NUMERICAL RESULTS

We report in this section preliminary results for the moving horizon estimator (12) as well as its refinement with a few iterations of a gradient-based method. We employ extended and unscented Kalman filters as benchmark given their similar computational complexity\(^2\). We use the data set test10-500 available online at http://www.stanford.edu/~yyue/ in which the sensor nodes are randomly distributed in the unit box \([-0.5, 0.5]^2\) in 2 dimensions. For a quantitative comparison, we run all our algorithms for a number of independent Monte-Carlo trials and we consider the positioning root mean squared error (PRMSE) of the algorithms at the \(\tau\)-th discrete time step:

\[
\text{PRMSE}_\tau := \sqrt{\frac{1}{R} \sum_{i=1}^{R} \sum_{\tau=1}^{N} \epsilon_{i,r,\tau}^2 / R_i}
\]

where \(R\) is the number of Monte-Carlo trials and \(\epsilon_{i,r,\tau}\) is the distance between the real location of the \(i\)-th node and its estimated location at the \(\tau\)-th trial of the discrete time step \(\tau\). (Note that \(\text{PRMSE}_\tau\) is not averaged over the sensor nodes).

The other parameters of the simulated scenario are the number of sensors \(n = 30\), number of anchors \(m = 6\), state matrices \(\Phi_i = (I_2, 0.1I_2; 0_2, I_2)\), measurement noise standard deviation \(\sigma_i = \sigma_{i,k,a} = 0.05\), process noise variance \(\Sigma_w = \sigma_w^2 I_1\) with \(\sigma_w = 5e-3\), initial condition \(\hat{s}_{i,0} = (0, 0, 0, 0)^T\) and variance \(\Pi_{s_i,0} = 1/12I_4\), time window \(T = 5\), and number of Monte-Carlo trials \(R = 10\). The number of range measurements per sensor node is limited to 3.

Figure 1 shows the PRMSE\(_\tau\) of the estimators with respect to the discrete time, as well as the posteriori Cramér Rao bound (PCRB) \(^8\) and the performance of the edge-based static convex relaxation of [2] applied on the range measurements alone at each time step. As we can see, the proposed estimator (12) performs better in terms of convergence speed, while the accuracy suffers from its relaxed nature. Nonetheless, the proposed refinement helps in restoring a similar steady-state accuracy than the other standard filters, without a significant increase in computational complexity.

VIII. CONCLUSIONS

We have proposed a moving horizon estimator for a mobile sensor network localization problem. The estimator has been derived from a convex relaxation of the MAP formulation of the problem at hand. As for its properties, it has a comparable computational complexity with the more standard extended

\(^1\)Computational complexity. The most demanding operations in the EKF are matrix inversions of dimension \(n\) and \(E_r\), for the covariance of process noise and measurement noise, leading to \(O(\max\{n^3, E_r^3\})\). UKF’s most demanding operations are matrix inversions of size equal to the augmented state, i.e., \(2n\) in prediction and \(n + E_r\) in updating, leading to \(O((n + E_r)^3)\). Computational complexity of a SDP is reported in [13]. The flop count of the edge-based heuristic is dependent on how the specific solver handles sparse matrices; a lower bound on the count is \(O(T n^2 E_r)\) [12], [13].

\(^2\)A comparison with the more computational demanding particle filters is currently ongoing work and will be presented in future publications.

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